A Murray-von Neumann type classification of C*-algebras

黃毅青 Ngai-Ching Wong

Department of Applied Mathematics
National Sun Yat-sen University
Taiwan

台灣　中山大學　應用數學系

Jointed with Chi-Keung NG 吳志强 (Nankai Univ.)

C*-algebras and W*-algebras

A C*-algebra is a (complex) B-space A with multiplication and involution satisfying that

C*-algebras and W*-algebras

A C*-algebra is a (complex) B-space A with multiplication and involution satisfying that
(1) $\|ab\| \leq \|a\|\|b\|, \forall a, b \in A$.

C*-algebras and W*-algebras

A C*-algebra is a (complex) B-space A with multiplication and involution satisfying that
(1) $\|ab\| \leq \|a\|\|b\|, \forall a, b \in A$.
(2) $\|a^*a\| = \|a\|^2, \forall a \in A$.

C*-algebras and W*-algebras

A C*-algebra is a (complex) B-space A with multiplication and involution satisfying that
(1) $\|ab\| \leq \|a\|\|b\|, \forall a, b \in A$.
(2) $\|a^*a\| = \|a\|^2, \forall a \in A$.

A W*-algebra is a C*-alg M with a predual M_*.

C*-algebras and W*-algebras

A C*-algebra is a (complex) B-space A with multiplication and involution satisfying that
(1) $\|ab\| \leq \|a\|\|b\|, \forall a, b \in A$.
(2) $\|a^*a\| = \|a\|^2, \forall a \in A$.

A W*-algebra is a C*-alg M with a predual M_*.

Gelfand-Naimark-Segal Theorem.

(a) Every abelian C*-alg $A \cong C_0(X)$ (W*-alg $M \cong C(\Omega)$) for some loc cpt Hausdorff X (hyperstonian Ω).

C*-algebras and W*-algebras

A C*-algebra is a (complex) B-space A with multiplication and involution satisfying that
(1) $\|ab\| \leq \|a\|\|b\|, \forall a, b \in A$.
(2) $\|a^*a\| = \|a\|^2, \forall a \in A$.

A W*-algebra is a C*-alg M with a predual M_*.

Gelfand-Naimark-Segal Theorem.

(a) Every abelian C*-alg $A \cong C_0(X)$ (W*-alg $M \cong C(\Omega)$) for some loc cpt Hausdorff X (hyperstonian Ω).

(b) Every C*-alg A (Every W*-alg M) $\hookrightarrow B(H)$ as a norm (weak*) closed *-subalg for some H-space H.

C*-algebras and W*-algebras

A C*-algebra is a (complex) B-space A with multiplication and involution satisfying that
(1) $\|ab\| \leq \|a\|\|b\|, \forall a, b \in A$.
(2) $\|a^*a\| = \|a\|^2, \forall a \in A$.

A W*-algebra is a C*-alg M with a predual M_*.

Gelfand-Naimark-Segal Theorem.

(a) Every abelian C*-alg $A \cong C_0(X)$ (W*-alg $M \cong C(\Omega)$) for some loc cpt Hausdorff X (hyperstonian Ω).

(b) Every C*-alg A (Every W*-alg M) $\hookrightarrow B(H)$ as a norm (weak*) closed *-subalg for some H-space H.

Classifications of W*-algebras

Muarry-von Nuemann classification scheme:

Classifications of W*-algebras

Muarry-von Nuemann classification scheme:

p is a projection in a W*-algebra M if $p = p^2 = p^*$.

Classifications of W*-algebras

Muarry-von Nuemann classification scheme:

p is a projection in a W*-algebra M if $p = p^2 = p^*$.

A subprojection $q \leq p$ if $qp = q$.

Classifications of W*-algebras

Muarry-von Nuemann classification scheme:

p is a projection in a W*-algebra M if $p = p^2 = p^*$.

A subprojection $q \leq p$ if $qp = q$.

u in M is a partial isometry if both u^*u, uu^* are projections.

Classifications of W*-algebras

Muarry-von Nuemann classification scheme:

p is a projection in a W*-algebra M if $p = p^2 = p^*$.

A subprojection $q \leq p$ if $qp = q$.

u in M is a partial isometry if both u^*u, uu^* are projections.

Two projection p, q are equivalent, written $p \sim q$, if there is a partial isometry u in M such that

$$p = uu^* \quad q = u^*u.$$

Classifications of W*-algebras

Muarry-von Nuemann classification scheme:

p is a projection in a W*-algebra M if $p = p^2 = p^*$.

A subprojection $q \leq p$ if $qp = q$.

u in M is a partial isometry if both u^*u, uu^* are projections.

Two projection p, q are equivalent, written $p \sim q$, if there is a partial isometry u in M such that

$$p = uu^* \quad q = u^*u.$$

A projection p in a W*-algebra M is

abelian if pMp is abelian;

A projection p in a W*-algebra M is

abelian if pMp is abelian;

finite if $q \le p$ with $q \sim p \implies q = p$;

A projection p in a W*-algebra M is

abelian if pMp is abelian;
finite if $q \leq p$ with $q \sim p \implies q = p$;
infinite if p is not finite;

A projection p in a W*-algebra M is

abelian if pMp is abelian;

finite if $q \leq p$ with $q \sim p \implies q = p$;

infinite if p is not finite;

purely infinite if p contains no nonzero finite subproj.

A projection p in a W*-algebra M is

abelian if pMp is abelian;

finite if $q \leq p$ with $q \sim p \implies q = p$;

infinite if p is not finite;

purely infinite if p contains no nonzero finite subproj.

Clearly, abelian projections are finite.

A projection p in a W*-algebra M is

abelian if pMp is abelian;

finite if $q \leq p$ with $q \sim p \implies q = p$;

infinite if p is not finite;

purely infinite if p contains no nonzero finite subproj.

Clearly, abelian projections are finite.

A proj z in the center of M is a central proj.

A proj z in the center of M is a central proj.

- $z_{\text{fin}} = \sup\{$ finite central proj in $M\}$,

A proj z in the center of M is a central proj.

- $z_{\text{fin}} = \sup\{$ finite central proj in $M\}$,

Call M

- finite or properly infinite if $z_{\text{fin}} = 1$ or 0.

A proj z in the center of M is a central proj.

- $z_{\text{fin}} = \sup\{$ finite central proj in $M\}$,
- $z_3 = \sup\{$ purely infinite central proj in $M\}$.

Call M

- finite or properly infinite if $z_{\text{fin}} = 1$ or 0.

A proj z in the center of M is a central proj.

- $z_{\text{fin}} = \sup\{$ finite central proj in $M\}$,
- $z_3 = \sup\{$ purely infinite central proj in $M\}$.

Call M

- finite or properly infinite if $z_{\text{fin}} = 1$ or 0.
- purely infinite or semi-finite if $z_3 = 1$ or 0.

Types of W*-algebras

A W*-algebra M is

- Type I if every nonzero central proj. contains a nonzero abelian subproj.;

Types of W*-algebras

A W*-algebra M is

Type I if every nonzero central proj. contains a nonzero abelian subproj.;

Type II if every nonzero central proj. contains a nonzero finite subproj., but no nonzero abelian subproj.;

Types of W*-algebras

A W*-algebra M is

Type I if every nonzero central proj. contains a nonzero abelian subproj.;

Type II if every nonzero central proj. contains a nonzero finite subproj., but no nonzero abelian subproj.;

Type III if M is purely infinite, or $z_3 = 1$, i.e., there is no nonzero finite proj. in M.

Types of W*-algebras

A W*-algebra M is

- Type I if every nonzero central proj. contains a nonzero abelian subproj.;
- Type II if every nonzero central proj. contains a nonzero finite subproj., but no nonzero abelian subproj.;
- Type III if M is purely infinite, or $z_3 = 1$, i.e., there is no nonzero finite proj. in M.

Type decomposition of a W*-alg

$$M = M_{11} \oplus M_{1\infty} \oplus M_{21} \oplus M_{2\infty} \oplus M_3,$$

Type decomposition of a W*-alg

$$M = M_{11} \oplus M_{1\infty} \oplus M_{21} \oplus M_{2\infty} \oplus M_3,$$

where

M_{11} is finite Type I, $M_{1\infty}$ is properly infinite Type I,

Type decomposition of a W*-alg

$$M = M_{11} \oplus M_{1\infty} \oplus M_{21} \oplus M_{2\infty} \oplus M_3,$$

where

M_{11} is finite Type I, $M_{1\infty}$ is properly infinite Type I,
M_{21} is finite Type II, $M_{2\infty}$ is properly infinite Type II,

Type decomposition of a W*-alg

$$M = M_{11} \oplus M_{1\infty} \oplus M_{21} \oplus M_{2\infty} \oplus M_3,$$

where

M_{11} is finite Type I, $M_{1\infty}$ is properly infinite Type I, M_{21} is finite Type II, $M_{2\infty}$ is properly infinite Type II, and M_3 is purely infinite.

Type decomposition of a W*-alg

$$M = M_{11} \oplus M_{1\infty} \oplus M_{21} \oplus M_{2\infty} \oplus M_3,$$

where

M_{11} is finite Type I, $M_{1\infty}$ is properly infinite Type I, M_{21} is finite Type II, $M_{2\infty}$ is properly infinite Type II, and M_3 is purely infinite.

In this talk, we shall develop in parallel a type classification of C*-algebras.

Type decomposition of a W*-alg

$$M = M_{11} \oplus M_{1\infty} \oplus M_{21} \oplus M_{2\infty} \oplus M_3,$$

where

M_{11} is finite Type I, $M_{1\infty}$ is properly infinite Type I, M_{21} is finite Type II, $M_{2\infty}$ is properly infinite Type II, and M_3 is purely infinite.

In this talk, we shall develop in parallel a type classification of C*-algebras.

The main difficulty is that a C*-algebra A can have no nonzero projection at all.

Type decomposition of a W*-alg

$$M = M_{11} \oplus M_{1\infty} \oplus M_{21} \oplus M_{2\infty} \oplus M_3,$$

where

M_{11} is finite Type I, $M_{1\infty}$ is properly infinite Type I, M_{21} is finite Type II, $M_{2\infty}$ is properly infinite Type II, and M_3 is purely infinite.

In this talk, we shall develop in parallel a type classification of C*-algebras.

The main difficulty is that a C*-algebra A can have no nonzero projection at all.

Resolve: Use “open projections".

Type decomposition of a W*-alg

$$M = M_{11} \oplus M_{1\infty} \oplus M_{21} \oplus M_{2\infty} \oplus M_3,$$

where

M_{11} is finite Type I, $M_{1\infty}$ is properly infinite Type I, M_{21} is finite Type II, $M_{2\infty}$ is properly infinite Type II, and M_3 is purely infinite.

In this talk, we shall develop in parallel a type classification of C*-algebras.

The main difficulty is that a C*-algebra A can have no nonzero projection at all.

Resolve: Use “open projections".

Open projs and hered. subalgs

Let A be a C*-alg with bidual W*-alg A^{**}.

Open projs and hered. subalgs

Let A be a C*-alg with bidual W*-alg A^{**}.

A proj. $p \in A^{**}$ is an open proj. of A (Akemann '69) if $\exists a_\lambda \in A_+$ with $a_\lambda \uparrow p$ in $\sigma(A^{**}, A^*)$-top.

Open projs and hered. subalgs

Let A be a C*-alg with bidual W*-alg A^{**}.

A proj. $p \in A^{**}$ is an open proj. of A (Akemann '69) if $\exists a_\lambda \in A_+$ with $a_\lambda \uparrow p$ in $\sigma(A^{**}, A^*)$-top.

When $A = C_0(X)$ is abelian, open proj. of A are exactly characteristic functions χ_U of open subsets U of the spectrum X of A.

Open projs and hered. subalgs

Let A be a C*-alg with bidual W*-alg A^{**}.

A proj. $p \in A^{**}$ is an open proj. of A (Akemann '69) if $\exists a_\lambda \in A_+$ with $a_\lambda \uparrow p$ in $\sigma(A^{**}, A^*)$-top.

When $A = C_0(X)$ is abelian, open proj. of A are exactly characteristic functions χ_U of open subsets U of the spectrum X of A.

In general, open projections p
$\longleftrightarrow$ hereditary C^*-subalgs $\mathrm{her}(p) := pA^{**}p \cap A$ of A.

Open projs and hered. subalgs

Let A be a C*-alg with bidual W*-alg A^{**}.

A proj. $p \in A^{**}$ is an open proj. of A (Akemann '69) if $\exists a_\lambda \in A_+$ with $a_\lambda \uparrow p$ in $\sigma(A^{**}, A^*)$-top.

When $A = C_0(X)$ is abelian, open proj. of A are exactly characteristic functions χ_U of open subsets U of the spectrum X of A.

In general, open projections p
$\longleftrightarrow$ hereditary C^*-subalgs $\mathrm{her}(p) := pA^{**}p \cap A$ of A.

Moreover, central open projections z
$\longleftrightarrow$ closed ideals $\mathrm{her}(z) = zA^{**} \cap A$.

Open projs and hered. subalgs

Let A be a C*-alg with bidual W*-alg A^{**}.

A proj. $p \in A^{**}$ is an open proj. of A (Akemann '69) if $\exists a_\lambda \in A_+$ with $a_\lambda \uparrow p$ in $\sigma(A^{**}, A^*)$-top.

When $A = C_0(X)$ is abelian, open proj. of A are exactly characteristic functions χ_U of open subsets U of the spectrum X of A.

In general, open projections p
$\longleftrightarrow$ hereditary C^*-subalgs $\mathrm{her}(p) := pA^{**}p \cap A$ of A.

Moreover, central open projections z
$\longleftrightarrow$ closed ideals $\mathrm{her}(z) = zA^{**} \cap A$.

Open projs as open sets

Denote by $\mathrm{OP}(A)$ the family of all open projs $p \in A^{**}$ of a C*-alg. A.

Open projs as open sets

Denote by $\mathrm{OP}(A)$ the family of all open projs $p \in A^{**}$ of a C*-alg. A.

Proposition. Suppose that A and B are C^*-algs.

Open projs as open sets

Denote by $\mathrm{OP}(A)$ the family of all open projs $p \in A^{**}$ of a C*-alg. A.

Proposition. Suppose that A and B are C^*-algs.

1. If $\Phi : A^{**} \to B^{**}$ is a *-isom s.t. $\Phi(\mathrm{OP}(A)) = \mathrm{OP}(B)$, then $\Phi(A) = B$.

Open projs as open sets

Denote by $\mathrm{OP}(A)$ the family of all open projs $p \in A^{**}$ of a C*-alg. A.

Proposition. Suppose that A and B are C^*-algs.

1. If $\Phi : A^{**} \to B^{**}$ is a *-isom s.t. $\Phi(\mathrm{OP}(A)) = \mathrm{OP}(B)$, then $\Phi(A) = B$.

2. If B is a hereditary C*-subalg. of A, then $\mathrm{OP}(B) = \mathrm{OP}(A) \cap B^{**}$.

Open projs as open sets

Denote by $\mathrm{OP}(A)$ the family of all open projs $p \in A^{**}$ of a C*-alg. A.

Proposition. Suppose that A and B are C^*-algs.

1. If $\Phi : A^{**} \to B^{**}$ is a *-isom s.t. $\Phi(\mathrm{OP}(A)) = \mathrm{OP}(B)$, then $\Phi(A) = B$.

2. If B is a hereditary C*-subalg. of A, then $\mathrm{OP}(B) = \mathrm{OP}(A) \cap B^{**}$.

Equivalences among open projs

We now consider a suitable equivalence relation on $\mathrm{OP}(A)$.

Equivalences among open projs

We now consider a suitable equivalence relation on $\mathrm{OP}(A)$.

A naive choice is to use the original “Murray-von Neumann equivalence” $\sim_{\mathrm{Mv}}$.

Equivalences among open projs

We now consider a suitable equivalence relation on $\mathrm{OP}(A)$.

A naive choice is to use the original “Murray-von Neumann equivalence” $\sim_{\mathrm{Mv}}$.

However, this choice is not good because (H. Lin ’90) tell us that two open projs

$$p \sim_{\mathrm{Mv}} q \quad \not\Rightarrow \quad \mathrm{her}(p) \cong \mathrm{her}(q) \text{ as C*-algs.}$$

Equivalences among open projs

We now consider a suitable equivalence relation on $\mathrm{OP}(A)$.

A naive choice is to use the original "Murray-von Neumann equivalence" $\sim_{\mathrm{Mv}}$.

However, this choice is not good because (H. Lin '90) tell us that two open projs

$$p \sim_{\mathrm{Mv}} q \quad \not\Rightarrow \quad \mathrm{her}(p) \cong \mathrm{her}(q) \text{ as C*-algs.}$$

On the other hand, one might define two open projs $p \sim_{\mathrm{her}} q$ whenever $\mathrm{her}(p) \cong \mathrm{her}(q)$.

On the other hand, one might define two open projs $p \sim_{\mathrm{her}} q$ whenever $\mathrm{her}(p) \cong \mathrm{her}(q)$.

The problem of this choice is that two distinct open projs $\chi_U \neq \chi_V$ of $C([0,1])$ can be equivalent (if U, V are homeomorphic open subsets of $[0,1]$).

On the other hand, one might define two open projs $p \sim_{\mathrm{her}} q$ whenever $\mathrm{her}(p) \cong \mathrm{her}(q)$.

The problem of this choice is that two distinct open projs $\chi_U \neq \chi_V$ of $C([0,1])$ can be equivalent (if U, V are homeomorphic open subsets of $[0,1]$).

This means that the resulting classification, even if possible, will be very different from the Murray-von Neumann classification.

On the other hand, one might define two open projs $p \sim_{\mathrm{her}} q$ whenever $\mathrm{her}(p) \cong \mathrm{her}(q)$.

The problem of this choice is that two distinct open projs $\chi_U \neq \chi_V$ of $C([0,1])$ can be equivalent (if U, V are homeomorphic open subsets of $[0,1]$).

This means that the resulting classification, even if possible, will be very different from the Murray-von Neumann classification.

After some thoughts, we end up with an equivalence relation.
Two open proj. $p \sim_{\rm sp} q$ if $\exists$ partial isom $v \in A^{**}$ s.t.

$$v^* \operatorname{her}_A(p) v = \operatorname{her}_A(q) \quad \text{and} \quad v \operatorname{her}_A(q) v^* = \operatorname{her}_A(p).$$

After some thoughts, we end up with an equivalence relation.
Two open proj. $p \sim_{\text{sp}} q$ if $\exists$ partial isom $v \in A^{**}$ s.t.

$$v^* \operatorname{her}_A(p) v = \operatorname{her}_A(q) \quad \text{and} \quad v \operatorname{her}_A(q) v^* = \operatorname{her}_A(p).$$

Note that this relation is precisely the "hereditarily stable version" of the Murray-von Neumann equiv,

After some thoughts, we end up with an equivalence relation.
Two open proj. $p \sim_{\text{sp}} q$ if $\exists$ partial isom $v \in A^{**}$ s.t.

$$v^* \operatorname{her}_A(p) v = \operatorname{her}_A(q) \quad \text{and} \quad v \operatorname{her}_A(q) v^* = \operatorname{her}_A(p).$$

Note that this relation is precisely the “hereditarily stable version” of the Murray-von Neumann equiv, and it is stronger than both $\sim_{\text{Mv}}$ and $\sim_{\text{her}}$.

After some thoughts, we end up with an equivalence relation.
Two open proj. $p \sim_{\text{sp}} q$ if $\exists$ partial isom $v \in A^{**}$ s.t.

$$v^* \, \mathrm{her}_A(p) v \;=\; \mathrm{her}_A(q) \quad \text{and} \quad v \, \mathrm{her}_A(q) v^* \;=\; \mathrm{her}_A(p).$$

Note that this relation is precisely the "hereditarily stable version" of the Murray-von Neumann equiv, and it is stronger than both $\sim_{\text{Mv}}$ and $\sim_{\text{her}}$.

If M is a W*-alg and p, q are proj. in M, then

$$p \sim_{\text{sp}} q \quad \Leftrightarrow \quad p \sim_{\text{Mv}} q.$$

After some thoughts, we end up with an equivalence relation.
Two open proj. $p \sim_{\text{sp}} q$ if $\exists$ partial isom $v \in A^{**}$ s.t.

$$v^* \operatorname{her}_A(p) v = \operatorname{her}_A(q) \quad \text{and} \quad v \operatorname{her}_A(q) v^* = \operatorname{her}_A(p).$$

Note that this relation is precisely the “hereditarily stable version” of the Murray-von Neumann equiv, and it is stronger than both $\sim_{\text{Mv}}$ and $\sim_{\text{her}}$.

If M is a W*-alg and p, q are proj. in M, then

$$p \sim_{\text{sp}} q \quad \Leftrightarrow \quad p \sim_{\text{Mv}} q.$$

Peligrad and Zsidó ('00) introduced another equivalence relation on projections in A^{**}.

Peligrad and Zsidó ('00) introduced another equivalence relation on projections in A^{**}.

$p \sim_{\mathrm{PZ}} q$ if there is a partial isometry $v \in A^{**}$ such that

$p = vv^*, q = v^*v, v^* \operatorname{her}_A(p) \subseteq A$ and $v \operatorname{her}_A(q) \subseteq A$.

Peligrad and Zsidó ('00) introduced another equivalence relation on projections in A^{**}.

$p \sim_{\mathrm{PZ}} q$ if there is a partial isometry $v \in A^{**}$ such that

$p = vv^*, q = v^*v, v^* \operatorname{her}_A(p) \subseteq A$ and $v \operatorname{her}_A(q) \subseteq A$.

It is not difficult to see that $\sim_{\mathrm{PZ}}$ is stronger than $\sim_{\mathrm{sp}}$.

Peligrad and Zsidó ('00) introduced another equivalence relation on projections in A^{**}.

$p \sim_{\mathrm{PZ}} q$ if there is a partial isometry $v \in A^{**}$ such that

$p = vv^*, q = v^*v, v^* \operatorname{her}_A(p) \subseteq A$ and $v \operatorname{her}_A(q) \subseteq A$.

It is not difficult to see that $\sim_{\mathrm{PZ}}$ is stronger than $\sim_{\mathrm{sp}}$.

However, we decide to use $\sim_{\mathrm{sp}}$ as it seems to be more natural in the way of using open projections.

Peligrad and Zsidó ('00) introduced another equivalence relation on projections in A^{**}.

$p \sim_{\text{PZ}} q$ if there is a partial isometry $v \in A^{**}$ such that

$p = vv^*, q = v^*v, v^* \operatorname{her}_A(p) \subseteq A$ and $v \operatorname{her}_A(q) \subseteq A$.

It is not difficult to see that $\sim_{\text{PZ}}$ is stronger than $\sim_{\text{sp}}$.

However, we decide to use $\sim_{\text{sp}}$ as it seems to be more natural in the way of using open projections.

Proposition

Let $p, q \in \mathrm{OP}(A)$. TFAE.

Proposition

Let $p, q \in \mathrm{OP}(A)$. TFAE.

1. $p = vv^*$, $q = v^*v$, $v^* \operatorname{her}_A(p) v = \operatorname{her}_A(q)$, and $v \operatorname{her}_A(q) v^* = \operatorname{her}_A(p)$ for a partial isom $v \in A^{**}$.

Proposition

Let $p, q \in \mathrm{OP}(A)$. TFAE.

1. $p = vv^*$, $q = v^*v$, $v^* \operatorname{her}_A(p)v = \operatorname{her}_A(q)$, and $v \operatorname{her}_A(q)v^* = \operatorname{her}_A(p)$ for a partial isom $v \in A^{**}$.
2. $\operatorname{her}(q) = u^* \operatorname{her}(p)u$ and $\operatorname{her}(p) = u \operatorname{her}(q)u^*$ for a partial isom $u \in A^{**}$.

Proposition

Let $p, q \in \mathrm{OP}(A)$. TFAE.

1. $p = vv^*$, $q = v^*v$, $v^* \operatorname{her}_A(p)v = \operatorname{her}_A(q)$, and $v \operatorname{her}_A(q)v^* = \operatorname{her}_A(p)$ for a partial isom $v \in A^{**}$.
2. $\operatorname{her}(q) = u^* \operatorname{her}(p)u$ and $\operatorname{her}(p) = u \operatorname{her}(q)u^*$ for a partial isom $u \in A^{**}$.
3. $\operatorname{her}(q) \subseteq u^* \operatorname{her}(p)u$ and $\operatorname{her}(p) \subseteq u \operatorname{her}(q)u^*$ for a partial isom $u \in A^{**}$.

Proposition

Let $p, q \in \mathrm{OP}(A)$. TFAE.

1. $p = vv^*$, $q = v^*v$, $v^* \operatorname{her}_A(p)v = \operatorname{her}_A(q)$, and $v \operatorname{her}_A(q)v^* = \operatorname{her}_A(p)$ for a partial isom $v \in A^{**}$.
2. $\operatorname{her}(q) = u^* \operatorname{her}(p)u$ and $\operatorname{her}(p) = u \operatorname{her}(q)u^*$ for a partial isom $u \in A^{**}$.
3. $\operatorname{her}(q) \subseteq u^* \operatorname{her}(p)u$ and $\operatorname{her}(p) \subseteq u \operatorname{her}(q)u^*$ for a partial isom $u \in A^{**}$.
4. $q \leq v^*v$ and $v \operatorname{her}(q)v^* = \operatorname{her}(p)$ for a partial isom $v \in A^{**}$.

Proposition

Let $p, q \in \mathrm{OP}(A)$. TFAE.

1. $p = vv^*$, $q = v^*v$, $v^* \operatorname{her}_A(p)v = \operatorname{her}_A(q)$, and $v \operatorname{her}_A(q)v^* = \operatorname{her}_A(p)$ for a partial isom $v \in A^{**}$.
2. $\operatorname{her}(q) = u^* \operatorname{her}(p)u$ and $\operatorname{her}(p) = u \operatorname{her}(q)u^*$ for a partial isom $u \in A^{**}$.
3. $\operatorname{her}(q) \subseteq u^* \operatorname{her}(p)u$ and $\operatorname{her}(p) \subseteq u \operatorname{her}(q)u^*$ for a partial isom $u \in A^{**}$.
4. $q \leq v^*v$ and $v \operatorname{her}(q)v^* = \operatorname{her}(p)$ for a partial isom $v \in A^{**}$.
5. $\exists$ a partial isom $w \in A^{**}$ s.t. $p = ww^*$ and

$$\{w^*rw : r \in \mathrm{OP}(A); r \leq p\} = \{s \in \mathrm{OP}(A) : s \leq q\}$$

Proposition

Let $p, q \in \mathrm{OP}(A)$. TFAE.

1. $p = vv^*$, $q = v^*v$, $v^* \operatorname{her}_A(p)v = \operatorname{her}_A(q)$, and $v \operatorname{her}_A(q)v^* = \operatorname{her}_A(p)$ for a partial isom $v \in A^{**}$.
2. $\operatorname{her}(q) = u^* \operatorname{her}(p)u$ and $\operatorname{her}(p) = u \operatorname{her}(q)u^*$ for a partial isom $u \in A^{**}$.
3. $\operatorname{her}(q) \subseteq u^* \operatorname{her}(p)u$ and $\operatorname{her}(p) \subseteq u \operatorname{her}(q)u^*$ for a partial isom $u \in A^{**}$.
4. $q \leq v^*v$ and $v \operatorname{her}(q)v^* = \operatorname{her}(p)$ for a partial isom $v \in A^{**}$.
5. $\exists$ a partial isom $w \in A^{**}$ s.t. $p = ww^*$ and

$$\{w^*rw : r \in \mathrm{OP}(A); r \leq p\} = \{s \in \mathrm{OP}(A) : s \leq q\}$$

A difficulty

One might attempt to define

$p \lesssim_{\text{sp}} q$ if there is $q_1 \in \mathrm{OP}(A)$ with $p \sim_{\text{sp}} q_1 \leq q$.

A difficulty

One might attempt to define

$p \precsim_{\mathrm{sp}} q$ if there is $q_1 \in \mathrm{OP}(A)$ with $p \sim_{\mathrm{sp}} q_1 \leq q$.

However, unlike the Murray-von Neumann equivalence situation,

$$p \precsim_{\mathrm{sp}} q \text{ and } q \precsim_{\mathrm{sp}} p \quad \not\Rightarrow \quad p \sim_{\mathrm{sp}} q.$$

A difficulty

One might attempt to define

$p \precsim_{\text{sp}} q$ if there is $q_1 \in \mathrm{OP}(A)$ with $p \sim_{\text{sp}} q_1 \leq q$.

However, unlike the Murray-von Neumann equivalence situation,

$$p \precsim_{\text{sp}} q \text{ and } q \precsim_{\text{sp}} p \quad \not\Rightarrow \quad p \sim_{\text{sp}} q.$$

finite and abelian open projs

finite and abelian open projs

Let $q \in \mathrm{OP}(A)$, and p a proj in $qA^{**}q$. The *closure of* p *in* q, denoted by $\bar{p}^q$, is the smallest closed proj of $\mathrm{her}(q)$ dominating p.

finite and abelian open projs

Let $q \in \mathrm{OP}(A)$, and p a proj in $qA^{**}q$. The *closure of p in q*, denoted by $\bar{p}^q$, is the smallest closed proj of $\mathrm{her}(q)$ dominating p.

Let $p, q \in \mathrm{OP}(A)$ with $p \leq q$. p is

finite and abelian open projs

Let $q \in \mathrm{OP}(A)$, and p a proj in $qA^{**}q$. The *closure of p in q*, denoted by $\bar{p}^q$, is the smallest closed proj of $\mathrm{her}(q)$ dominating p.

Let $p, q \in \mathrm{OP}(A)$ with $p \leq q$. p is

i. *dense in q* if $\bar{p}^q = q$;

finite and abelian open projs

Let $q \in \mathrm{OP}(A)$, and p a proj in $qA^{**}q$. The *closure of p in q*, denoted by $\bar{p}^q$, is the smallest closed proj of $\mathrm{her}(q)$ dominating p.

Let $p, q \in \mathrm{OP}(A)$ with $p \leq q$. p is

i. *dense in q* if $\bar{p}^q = q$;

ii. *abelian* if $\mathrm{her}(p)$ is abelian;

finite and abelian open projs

Let $q \in \mathrm{OP}(A)$, and p a proj in $qA^{**}q$. The *closure of* p *in* q, denoted by $\bar{p}^q$, is the smallest closed proj of $\mathrm{her}(q)$ dominating p.

Let $p, q \in \mathrm{OP}(A)$ with $p \leq q$. p is

i. *dense in* q if $\bar{p}^q = q$;

ii. *abelian* if $\mathrm{her}(p)$ is abelian;

iii. C^**-finite* if $\bar{r}^s = s$ whenever $r, s \in \mathrm{OP}(\mathrm{her}(p))$ with $r \leq s$ and $r \sim_{\text{sp}} s$.

finite and abelian open projs

Let $q \in \mathrm{OP}(A)$, and p a proj in $qA^{**}q$. The *closure of* p *in* q, denoted by $\bar{p}^q$, is the smallest closed proj of $\mathrm{her}(q)$ dominating p.

Let $p, q \in \mathrm{OP}(A)$ with $p \leq q$. p is

i. *dense in* q if $\bar{p}^q = q$;

ii. *abelian* if $\mathrm{her}(p)$ is abelian;

iii. C^**-finite* if $\bar{r}^s = s$ whenever $r, s \in \mathrm{OP}(\mathrm{her}(p))$ with $r \leq s$ and $r \sim_{\text{sp}} s$.

If p is dense in q, then $\mathrm{her}(p)$ is *essential* in $\mathrm{her}(q)$.

finite and abelian open projs

Let $q \in \mathrm{OP}(A)$, and p a proj in $qA^{**}q$. The *closure of p in q*, denoted by $\bar{p}^q$, is the smallest closed proj of $\mathrm{her}(q)$ dominating p.

Let $p, q \in \mathrm{OP}(A)$ with $p \leq q$. p is

i. *dense in q* if $\bar{p}^q = q$;

ii. *abelian* if $\mathrm{her}(p)$ is abelian;

iii. *C^*-finite* if $\bar{r}^s = s$ whenever $r, s \in \mathrm{OP}(\mathrm{her}(p))$ with $r \leq s$ and $r \sim_{\text{sp}} s$.

If p is dense in q, then $\mathrm{her}(p)$ is *essential* in $\mathrm{her}(q)$.

Let $p \in \mathrm{OP}(A)$.

Let $p \in \mathrm{OP}(A)$.

(a) Suppose that p is abelian. If $r, s \in \mathrm{OP}(\mathrm{her}(p))$ satisfying $r \leq s$ and $r \sim_{\mathrm{sp}} s$, then $r = s$. Thus, p is C^*-finite.

Let $p \in \mathrm{OP}(A)$.

(a) Suppose that p is abelian. If $r, s \in \mathrm{OP}(\mathrm{her}(p))$ satisfying $r \leq s$ and $r \sim_{\text{sp}} s$, then $r = s$. Thus, p is C^*-finite.

(b) If $\mathrm{her}(p)$ is finite dimensional, then p is C^*-finite.

Let $p \in \mathrm{OP}(A)$.

(a) Suppose that p is abelian. If $r, s \in \mathrm{OP}(\mathrm{her}(p))$ satisfying $r \leq s$ and $r \sim_{\mathrm{sp}} s$, then $r = s$. Thus, p is C^*-finite.

(b) If $\mathrm{her}(p)$ is finite dimensional, then p is C^*-finite.

(c) One might ask why not define C^*-finiteness of p as: for any open subproj $r \leq p$ with $r \sim_{\mathrm{sp}} p$, one has $\bar{r}^p = p$.

Let $p \in \mathrm{OP}(A)$.

(a) Suppose that p is abelian. If $r, s \in \mathrm{OP}(\mathrm{her}(p))$ satisfying $r \leq s$ and $r \sim_{\mathrm{sp}} s$, then $r = s$. Thus, p is C^*-finite.

(b) If $\mathrm{her}(p)$ is finite dimensional, then p is C^*-finite.

(c) One might ask why not define C^*-finiteness of p as: for any open subproj $r \leq p$ with $r \sim_{\mathrm{sp}} p$, one has $\bar{r}^p = p$.

The reason is that the stronger condition above ensures open subprojs of a C^*-finite proj being C^*-finite.

Let $p \in \mathrm{OP}(A)$.

(a) Suppose that p is abelian. If $r, s \in \mathrm{OP}(\mathrm{her}(p))$ satisfying $r \leq s$ and $r \sim_{\mathrm{sp}} s$, then $r = s$. Thus, p is C^*-finite.

(b) If $\mathrm{her}(p)$ is finite dimensional, then p is C^*-finite.

(c) One might ask why not define C^*-finiteness of p as: for any open subproj $r \leq p$ with $r \sim_{\mathrm{sp}} p$, one has $\bar{r}^p = p$.

The reason is that the stronger condition above ensures open subprojs of a C^*-finite proj being C^*-finite.
Such a phenomena is automatic for W*-algs.

Let $p \in \mathrm{OP}(A)$.

(a) Suppose that p is abelian. If $r, s \in \mathrm{OP}(\mathrm{her}(p))$ satisfying $r \leq s$ and $r \sim_{\mathrm{sp}} s$, then $r = s$. Thus, p is C^*-finite.

(b) If $\mathrm{her}(p)$ is finite dimensional, then p is C^*-finite.

(c) One might ask why not define C^*-finiteness of p as: for any open subproj $r \leq p$ with $r \sim_{\mathrm{sp}} p$, one has $\bar{r}^p = p$.

The reason is that the stronger condition above ensures open subprojs of a C^*-finite proj being C^*-finite.
Such a phenomena is automatic for W*-algs.

Types of C*-algs

A C^*-alg A is said to be:

Types of C*-algs

A C^*-alg A is said to be:

i. C^*-*finite* if 1 is C*-finite;

Types of C*-algs

A C^*-alg A is said to be:

i. C^*-*finite* if 1 is C*-finite;

ii. C^*-*semi-finite* if every nonzero open proj dominates a nonzero C*-finite open proj;

Types of C*-algs

A C^*-alg A is said to be:

i. *C^*-finite* if 1 is C*-finite;

ii. *C^*-semi-finite* if every nonzero open proj dominates a nonzero C*-finite open proj;

iii. *of Type* $\mathfrak{A}$ if every nonzero central open proj dominates a nonzero abelian open proj;

Types of C*-algs

A C^*-alg A is said to be:

i. *C^*-finite* if 1 is C*-finite;

ii. *C^*-semi-finite* if every nonzero open proj dominates a nonzero C*-finite open proj;

iii. *of Type* $\mathfrak{A}$ if every nonzero central open proj dominates a nonzero abelian open proj;

iv. *of Type* $\mathfrak{B}$ if every nonzero central open proj dominates a nonzero C*-finite open projection, but no nonzero abelian open proj;

Types of C*-algs

A C^*-alg A is said to be:

i. *C^*-finite* if 1 is C*-finite;

ii. *C^*-semi-finite* if every nonzero open proj dominates a nonzero C*-finite open proj;

iii. *of Type* $\mathfrak{A}$ if every nonzero central open proj dominates a nonzero abelian open proj;

iv. *of Type* $\mathfrak{B}$ if every nonzero central open proj dominates a nonzero C*-finite open projection, but no nonzero abelian open proj;

v. *of Type* $\mathfrak{C}$ if there is no nonzero C*-finite open projections.

Types of C*-algs

A C^*-alg A is said to be:

i. *C^*-finite* if 1 is C*-finite;

ii. *C^*-semi-finite* if every nonzero open proj dominates a nonzero C*-finite open proj;

iii. *of Type* $\mathfrak{A}$ if every nonzero central open proj dominates a nonzero abelian open proj;

iv. *of Type* $\mathfrak{B}$ if every nonzero central open proj dominates a nonzero C*-finite open projection, but no nonzero abelian open proj;

v. *of Type* $\mathfrak{C}$ if there is no nonzero C*-finite open projections.

A C^*-algebra A is

A C^*-algebra A is

- C^*-finite iff for each hered C^*-subalg $B \subseteq A$, every hered C^*-subalg of B which is spatially isomorphic to B, is essential in B;

A C^*-algebra A is

- C^*-finite iff for each hered C^*-subalg $B \subseteq A$, every hered C^*-subalg of B which is spatially isomorphic to B, is essential in B;
- C^*-semi-finite iff every non-zero hered C^*-subalg of A contains a non-zero C^*-finite hered C^*-subalg;

- of type $\mathfrak{A}$ iff every non-zero closed ideal of A contains a non-zero abelian hered C^*-subalg;

- of type $\mathfrak{A}$ iff every non-zero closed ideal of A contains a non-zero abelian hered C^*-subalg;

- of type $\mathfrak{B}$ iff A does not contain any non-zero abelian hered C^*-subalg and every non-zero closed ideal of A contains a non-zero C^*-finite hered C^*-subalg;

- of type $\mathfrak{A}$ iff every non-zero closed ideal of A contains a non-zero abelian hered C^*-subalg;
- of type $\mathfrak{B}$ iff A does not contain any non-zero abelian hered C^*-subalg and every non-zero closed ideal of A contains a non-zero C^*-finite hered C^*-subalg;
- of type $\mathfrak{C}$ iff A does not contain any non-zero C^*-finite hered C^*-subalg.

- of type $\mathfrak{A}$ iff every non-zero closed ideal of A contains a non-zero abelian hered C^*-subalg;
- of type $\mathfrak{B}$ iff A does not contain any non-zero abelian hered C^*-subalg and every non-zero closed ideal of A contains a non-zero C^*-finite hered C^*-subalg;
- of type $\mathfrak{C}$ iff A does not contain any non-zero C^*-finite hered C^*-subalg.

Let A be a general C*-algebra.

(a) A is of type $\mathfrak{A}$ iff $\exists$ abelian hered C^*-subalg of A generating an essential closed ideal of A.

Let A be a general C*-algebra.

(a) A is of type $\mathfrak{A}$ iff $\exists$ abelian hered C^*-subalg of A generating an essential closed ideal of A.

(b) A is C^*-semi-finite iff $\exists$ C^*-finite hered C^*-subalg of A generating an essential closed ideal of A.

Let A be a general C*-algebra.

(a) A is of type $\mathfrak{A}$ iff $\exists$ abelian hered C^*-subalg of A generating an essential closed ideal of A.

(b) A is C^*-semi-finite iff $\exists$ C^*-finite hered C^*-subalg of A generating an essential closed ideal of A.

Suppose that A is simple.

Suppose that A is simple.

(a) A is either of type $\mathfrak{A}$, type $\mathfrak{B}$ or type $\mathfrak{C}$.

Suppose that A is simple.

(a) A is either of type $\mathfrak{A}$, type $\mathfrak{B}$ or type $\mathfrak{C}$.

(b) A is of type $\mathfrak{A}$ if and only if A is of type I.

Suppose that A is simple.

(a) A is either of type $\mathfrak{A}$, type $\mathfrak{B}$ or type $\mathfrak{C}$.

(b) A is of type $\mathfrak{A}$ if and only if A is of type I.

(c) If A is of type II in the sense of (Cuntz and Pedersen '79), then A is of type $\mathfrak{B}$.

Suppose that A is simple.

(a) A is either of type $\mathfrak{A}$, type $\mathfrak{B}$ or type $\mathfrak{C}$.

(b) A is of type $\mathfrak{A}$ if and only if A is of type I.

(c) If A is of type II in the sense of (Cuntz and Pedersen '79), then A is of type $\mathfrak{B}$.

(d) If A is purely infinite in the sense of (Cuntz '81), then A is of type $\mathfrak{C}$.

Suppose that A is simple.

(a) A is either of type $\mathfrak{A}$, type $\mathfrak{B}$ or type $\mathfrak{C}$.

(b) A is of type $\mathfrak{A}$ if and only if A is of type I.

(c) If A is of type II in the sense of (Cuntz and Pedersen '79), then A is of type $\mathfrak{B}$.

(d) If A is purely infinite in the sense of (Cuntz '81), then A is of type $\mathfrak{C}$.

Let A be a C^*-algebra with real rank zero.

Let A be a C^*-algebra with real rank zero.

(a) A is of type $\mathfrak{A}$ iff every nonzero proj in A dominates a nonzero abelian proj in A.

Let A be a C^*-algebra with real rank zero.

(a) A is of type $\mathfrak{A}$ iff every nonzero proj in A dominates a nonzero abelian proj in A.

(b) A is of type $\mathfrak{B}$ iff every nonzero proj in A is non-abelian but dominates a nonzero C^*-finite proj in A.

Let A be a C^*-algebra with real rank zero.

(a) A is of type $\mathfrak{A}$ iff every nonzero proj in A dominates a nonzero abelian proj in A.

(b) A is of type $\mathfrak{B}$ iff every nonzero proj in A is non-abelian but dominates a nonzero C^*-finite proj in A.

(c) A is of type $\mathfrak{C}$ iff A does not contain any non-zero C^*-finite proj.

Let A be a C^*-algebra with real rank zero.

(a) A is of type $\mathfrak{A}$ iff every nonzero proj in A dominates a nonzero abelian proj in A.

(b) A is of type $\mathfrak{B}$ iff every nonzero proj in A is non-abelian but dominates a nonzero C^*-finite proj in A.

(c) A is of type $\mathfrak{C}$ iff A does not contain any non-zero C^*-finite proj.

(d) A is C^*-semi-finite iff every nonzero proj in A dominates a nonzero C^*-finite proj in A.

Let A be a C^*-algebra with real rank zero.

(a) A is of type $\mathfrak{A}$ iff every nonzero proj in A dominates a nonzero abelian proj in A.

(b) A is of type $\mathfrak{B}$ iff every nonzero proj in A is non-abelian but dominates a nonzero C^*-finite proj in A.

(c) A is of type $\mathfrak{C}$ iff A does not contain any non-zero C^*-finite proj.

(d) A is C^*-semi-finite iff every nonzero proj in A dominates a nonzero C^*-finite proj in A.

(a) Let A and B be two strongly Morita equivalent C^*-algs.

Then A is of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C*-smei-finite) iff B is of the same type.

(a) Let A and B be two strongly Morita equivalent C^*-algs.

Then A is of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C*-smei-finite) iff B is of the same type.

(b) A is C^*-semi-finite iff any non-zero closed ideal of A contains a nonzero C^*-finite hered C^*-subalg.

(a) Let A and B be two strongly Morita equivalent C^*-algs.

Then A is of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C*-smei-finite) iff B is of the same type.

(b) A is C^*-semi-finite iff any non-zero closed ideal of A contains a nonzero C^*-finite hered C^*-subalg.

(c) A is of type $\mathfrak{A}$ iff it is discrete, i.e.,

any non-zero open proj of A dominates a non-zero abelian open proj.

(a) Let A and B be two strongly Morita equivalent C^*-algs.

Then A is of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C*-smei-finite) iff B is of the same type.

(b) A is C^*-semi-finite iff any non-zero closed ideal of A contains a nonzero C^*-finite hered C^*-subalg.

(c) A is of type $\mathfrak{A}$ iff it is discrete, i.e.,

any non-zero open proj of A dominates a non-zero abelian open proj.

(d) A is of type $\mathfrak{B}$ iff it is anti-liminary and C^*-semi-finite.

(a) Let A and B be two strongly Morita equivalent C^*-algs.

Then A is of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C*-smei-finite) iff B is of the same type.

(b) A is C^*-semi-finite iff any non-zero closed ideal of A contains a nonzero C^*-finite hered C^*-subalg.

(c) A is of type $\mathfrak{A}$ iff it is discrete, i.e.,

any non-zero open proj of A dominates a non-zero abelian open proj.

(d) A is of type $\mathfrak{B}$ iff it is anti-liminary and C^*-semi-finite.

Hereditability of types

Let A be of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C^*-semi-finite).

Hereditability of types

Let A be of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C^*-semi-finite).

(a) If B is a hered C^*-subalg of A, then B is of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C^*-semi-finite).

Hereditability of types

Let A be of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C^*-semi-finite).

(a) If B is a hered C^*-subalg of A, then B is of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C^*-semi-finite).

(b) If A is a hered C^*-subalg of D, the closed ideal $I \subseteq D$ generated by A is of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C^*-semi-finite).

Hereditability of types

Let A be of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C^*-semi-finite).

(a) If B is a hered C^*-subalg of A, then B is of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C^*-semi-finite).

(b) If A is a hered C^*-subalg of D, the closed ideal $I \subseteq D$ generated by A is of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C^*-semi-finite).

(c) Suppose $1 \notin A$. Then the unitalization $\tilde{A}$ is of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C^*-semi-finite). The same is true for the multiplier alg $M(A)$.

Hereditability of types

Let A be of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C^*-semi-finite).

(a) If B is a hered C^*-subalg of A, then B is of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C^*-semi-finite).

(b) If A is a hered C^*-subalg of D, the closed ideal $I \subseteq D$ generated by A is of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C^*-semi-finite).

(c) Suppose $1 \notin A$. Then the unitalization $\tilde{A}$ is of type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, or C^*-semi-finite). The same is true for the multiplier alg $M(A)$.

Type $\mathfrak{A}$ algebras

(a) Any type I C^*-algebra is of type $\mathfrak{A}$.

Type $\mathfrak{A}$ algebras

(a) Any type I C^*-algebra is of type $\mathfrak{A}$.

(b) A is of type I iff every primitive quotient of A is of type $\mathfrak{A}$.

Type $\mathfrak{A}$ algebras

(a) Any type I C^*-algebra is of type $\mathfrak{A}$.

(b) A is of type I iff every primitive quotient of A is of type $\mathfrak{A}$.

(c) If A is of type $\mathfrak{A}$ and contains no essential primitive ideal, then A is of type I.

Type $\mathfrak{A}$ algebras

(a) Any type I C^*-algebra is of type $\mathfrak{A}$.

(b) A is of type I iff every primitive quotient of A is of type $\mathfrak{A}$.

(c) If A is of type $\mathfrak{A}$ and contains no essential primitive ideal, then A is of type I.

(d) If A is a simple C^*-alg of type $\mathfrak{A}$, then $A = \mathcal{K}(H)$ for some Hilbert space H. If, in addition, A is C^*-finite, then $A = M_n$ for some positive integer n.

Type $\mathfrak{A}$ algebras

(a) Any type I C^*-algebra is of type $\mathfrak{A}$.

(b) A is of type I iff every primitive quotient of A is of type $\mathfrak{A}$.

(c) If A is of type $\mathfrak{A}$ and contains no essential primitive ideal, then A is of type I.

(d) If A is a simple C^*-alg of type $\mathfrak{A}$, then $A = \mathcal{K}(H)$ for some Hilbert space H. If, in addition, A is C^*-finite, then $A = M_n$ for some positive integer n.

Type II and III algs

Following (Cuntz and Pedersen, ’79), $x \in A_+$ is *finite* if for any sequence $\{z_k\}$ in A with $x = \sum_{k=1}^{\infty} z_k^* z_k$ and $y = \sum_{k=1}^{\infty} z_k z_k^* \leq x$, one has $y = x$.

Type II and III algs

Following (Cuntz and Pedersen, ’79), $x \in A_+$ is *finite* if for any sequence $\{z_k\}$ in A with $x = \sum_{k=1}^{\infty} z_k^* z_k$ and $y = \sum_{k=1}^{\infty} z_k z_k^* \leq x$, one has $y = x$.

A is *finite* (resp, *semi-finite*) if every nonzero $x \in A_+$ is finite (resp, x dominates a non-zero finite element).

Type II and III algs

Following (Cuntz and Pedersen, '79), $x \in A_+$ is *finite* if for any sequence $\{z_k\}$ in A with $x = \sum_{k=1}^{\infty} z_k^* z_k$ and $y = \sum_{k=1}^{\infty} z_k z_k^* \leq x$, one has $y = x$.

A is *finite* (resp, *semi-finite*) if every nonzero $x \in A_+$ is finite (resp, x dominates a non-zero finite element).

Furthermore, A is of *type II* if it is anti-liminary and finite,

Type II and III algs

Following (Cuntz and Pedersen, ’79), $x \in A_+$ is *finite* if for any sequence $\{z_k\}$ in A with $x = \sum_{k=1}^{\infty} z_k^* z_k$ and $y = \sum_{k=1}^{\infty} z_k z_k^* \leq x$, one has $y = x$.

A is *finite* (resp, *semi-finite*) if every nonzero $x \in A_+$ is finite (resp, x dominates a non-zero finite element).

Furthermore, A is of *type II* if it is anti-liminary and finite,

while A is of *type III* if it has no non-zero finite element.

Type II and III algs

Following (Cuntz and Pedersen, ’79), $x \in A_+$ is *finite* if for any sequence $\{z_k\}$ in A with $x = \sum_{k=1}^{\infty} z_k^* z_k$ and $y = \sum_{k=1}^{\infty} z_k z_k^* \leq x$, one has $y = x$.

A is *finite* (resp, *semi-finite*) if every nonzero $x \in A_+$ is finite (resp, x dominates a non-zero finite element).

Furthermore, A is of *type II* if it is anti-liminary and finite,

while A is of *type III* if it has no non-zero finite element.

Type $\mathfrak{B}$ and C^*-semi-finite algs

(a) Infinite dim C^*-finite simple C^*-algs are of type $\mathfrak{B}$.

Type $\mathfrak{B}$ and C^*-semi-finite algs

(a) Infinite dim C^*-finite simple C^*-algs are of type $\mathfrak{B}$.

(b) If tracial states on A separates points of A_+, then A is C^*-finite.

Type $\mathfrak{B}$ and C^*-semi-finite algs

(a) Infinite dim C^*-finite simple C^*-algs are of type $\mathfrak{B}$.

(b) If tracial states on A separates points of A_+, then A is C^*-finite.

(c) If A is an infinite dim simple C^*-alg with a faithful tracial state, then A is of type $\mathfrak{B}$.

Type $\mathfrak{B}$ and C^*-semi-finite algs

(a) Infinite dim C^*-finite simple C^*-algs are of type $\mathfrak{B}$.

(b) If tracial states on A separates points of A_+, then A is C^*-finite.

(c) If A is an infinite dim simple C^*-alg with a faithful tracial state, then A is of type $\mathfrak{B}$.

(d) Every simple AF alg, not of the form $\mathcal{K}(H)$, is of type $\mathfrak{B}$.

Type $\mathfrak{B}$ and C^*-semi-finite algs

(a) Infinite dim C^*-finite simple C^*-algs are of type $\mathfrak{B}$.

(b) If tracial states on A separates points of A_+, then A is C^*-finite.

(c) If A is an infinite dim simple C^*-alg with a faithful tracial state, then A is of type $\mathfrak{B}$.

(d) Every simple AF alg, not of the form $\mathcal{K}(H)$, is of type $\mathfrak{B}$.

(e) If A is finite (resp, semi-finite, of type II), then A is C^*-finite (resp, C^*-semi-finite, of type $\mathfrak{B}$).

Type $\mathfrak{C}$ algebras

(a) If A is of type $\mathfrak{C}$, then it is of type III.

Type $\mathfrak{C}$ algebras

(a) If A is of type $\mathfrak{C}$, then it is of type III.

(b) If A has real rank zero and is purely infinite, then it is of type $\mathfrak{C}$.

Type $\mathfrak{C}$ algebras

(a) If A is of type $\mathfrak{C}$, then it is of type III.

(b) If A has real rank zero and is purely infinite, then it is of type $\mathfrak{C}$.

(c) If A is a separable purely infinite C^*-alg with stable rank one, then A is of type $\mathfrak{C}$.

Type $\mathfrak{C}$ algebras

(a) If A is of type $\mathfrak{C}$, then it is of type III.

(b) If A has real rank zero and is purely infinite, then it is of type $\mathfrak{C}$.

(c) If A is a separable purely infinite C^*-alg with stable rank one, then A is of type $\mathfrak{C}$.

(d) Any purely infinite C^*-algebra A is of type III.

The case of W*-algs

Let M be a W*-algebra.

(a) A proj $p \in M$ is finite iff it is C^*-finite.

The case of W*-algs

Let M be a W*-algebra.

(a) A proj $p \in M$ is finite iff it is C^*-finite.

(b) M is of type $\mathfrak{A}$ if and only if M is a type I von Neumann algebra.

The case of W*-algs

Let M be a W*-algebra.

(a) A proj $p \in M$ is finite iff it is C^*-finite.

(b) M is of type $\mathfrak{A}$ if and only if M is a type I von Neumann algebra.

(c) M is of type $\mathfrak{B}$ if and only if M is a type II von Neumann algebra.

The case of W*-algs

Let M be a W*-algebra.

(a) A proj $p \in M$ is finite iff it is C^*-finite.

(b) M is of type $\mathfrak{A}$ if and only if M is a type I von Neumann algebra.

(c) M is of type $\mathfrak{B}$ if and only if M is a type II von Neumann algebra.

(s) M is of type $\mathfrak{C}$ if and only if M is a type III von Neumann algebra.

The case of W*-algs

Let M be a W*-algebra.

(a) A proj $p \in M$ is finite iff it is C^*-finite.

(b) M is of type $\mathfrak{A}$ if and only if M is a type I von Neumann algebra.

(c) M is of type $\mathfrak{B}$ if and only if M is a type II von Neumann algebra.

(s) M is of type $\mathfrak{C}$ if and only if M is a type III von Neumann algebra.

(e) M is C^*-semi-finite if and only if M is a semi-finite von Neumann algebra.

The case of W*-algs

Let M be a W*-algebra.

(a) A proj $p \in M$ is finite iff it is C^*-finite.

(b) M is of type $\mathfrak{A}$ if and only if M is a type I von Neumann algebra.

(c) M is of type $\mathfrak{B}$ if and only if M is a type II von Neumann algebra.

(s) M is of type $\mathfrak{C}$ if and only if M is a type III von Neumann algebra.

(e) M is C^*-semi-finite if and only if M is a semi-finite von Neumann algebra.

Type factorizations

Let A be a C^*-algebra.

(a) $\exists$ a largest type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, and C^*-semi-finite) hered C^*-subalg $J_{\mathfrak{A}}$ (resp, $J_{\mathfrak{B}}$, $J_{\mathfrak{C}}$, and $J_{\mathfrak{sf}}$) of A, which is also an ideal of A.

Type factorizations

Let A be a C^*-algebra.

(a) $\exists$ a largest type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, and C^*-semi-finite) hered C^*-subalg $J_{\mathfrak{A}}$ (resp, $J_{\mathfrak{B}}$, $J_{\mathfrak{C}}$, and $J_{\mathfrak{sf}}$) of A, which is also an ideal of A.

Type factorizations

Let A be a C^*-algebra.

(a) $\exists$ a largest type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, and C^*-semi-finite) hered C^*-subalg $J_{\mathfrak{A}}$ (resp, $J_{\mathfrak{B}}$, $J_{\mathfrak{C}}$, and $J_{\mathfrak{sf}}$) of A, which is also an ideal of A.

(b) $J_{\mathfrak{A}}$, $J_{\mathfrak{B}}$ and $J_{\mathfrak{C}}$ are mutually disjoint s.t. $J_{\mathfrak{A}} + J_{\mathfrak{B}} + J_{\mathfrak{C}}$ is an essential closed ideal of A.

Type factorizations

Let A be a C^*-algebra.

(a) $\exists$ a largest type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, and C^*-semi-finite) hered C^*-subalg $J_{\mathfrak{A}}$ (resp, $J_{\mathfrak{B}}$, $J_{\mathfrak{C}}$, and $J_{\mathfrak{sf}}$) of A, which is also an ideal of A.

(b) $J_{\mathfrak{A}}$, $J_{\mathfrak{B}}$ and $J_{\mathfrak{C}}$ are mutually disjoint s.t. $J_{\mathfrak{A}} + J_{\mathfrak{B}} + J_{\mathfrak{C}}$ is an essential closed ideal of A.

If $e_{\mathfrak{A}}, e_{\mathfrak{B}}, e_{\mathfrak{C}}$ are central open projs in A^{**} with $J_{\mathfrak{A}} = \mathrm{her}(e_{\mathfrak{A}})$, $J_{\mathfrak{B}} = \mathrm{her}(e_{\mathfrak{B}})$ and $J_{\mathfrak{C}} = \mathrm{her}(e_{\mathfrak{C}})$, then

$$1 = \overline{e_{\mathfrak{A}} + e_{\mathfrak{B}}}^{1} + e_{\mathfrak{C}}.$$

Type factorizations

Let A be a C^*-algebra.

(a) $\exists$ a largest type $\mathfrak{A}$ (resp, $\mathfrak{B}$, $\mathfrak{C}$, and C^*-semi-finite) hered C^*-subalg $J_{\mathfrak{A}}$ (resp, $J_{\mathfrak{B}}$, $J_{\mathfrak{C}}$, and $J_{\mathfrak{sf}}$) of A, which is also an ideal of A.

(b) $J_{\mathfrak{A}}$, $J_{\mathfrak{B}}$ and $J_{\mathfrak{C}}$ are mutually disjoint s.t. $J_{\mathfrak{A}} + J_{\mathfrak{B}} + J_{\mathfrak{C}}$ is an essential closed ideal of A.

If $e_{\mathfrak{A}}, e_{\mathfrak{B}}, e_{\mathfrak{C}}$ are central open projs in A^{**} with $J_{\mathfrak{A}} = \mathrm{her}(e_{\mathfrak{A}})$, $J_{\mathfrak{B}} = \mathrm{her}(e_{\mathfrak{B}})$ and $J_{\mathfrak{C}} = \mathrm{her}(e_{\mathfrak{C}})$, then

$$1 = \overline{e_{\mathfrak{A}} + e_{\mathfrak{B}}}^{1} + e_{\mathfrak{C}}.$$

(c) $J_{\mathfrak{A}} + J_{\mathfrak{B}}$ is an essential closed ideal of $J_{\mathfrak{sf}}$.

If $e_{\mathfrak{sf}}$ is the central open proj of A with $J_{\mathfrak{sf}} = \mathrm{her}(e_{\mathfrak{sf}})$, then

$$e_{\mathfrak{sf}} = \overline{e_{\mathfrak{A}}}^{e_{\mathfrak{sf}}} + e_{\mathfrak{B}}.$$

(c) $J_{\mathfrak{A}} + J_{\mathfrak{B}}$ is an essential closed ideal of $J_{\mathfrak{sf}}$.

If $e_{\mathfrak{sf}}$ is the central open proj of A with $J_{\mathfrak{sf}} = \mathrm{her}(e_{\mathfrak{sf}})$, then

$$e_{\mathfrak{sf}} = \overline{e_{\mathfrak{A}}}^{e_{\mathfrak{sf}}} + e_{\mathfrak{B}}.$$

(d) $A/J^A_{\mathfrak{C}}$ is C^*-semi-finite and $A/(J^A_{\mathfrak{A}})^\perp$ is of type $\mathfrak{A}$.

(c) $J_{\mathfrak{A}} + J_{\mathfrak{B}}$ is an essential closed ideal of $J_{\mathfrak{sf}}$.

If $e_{\mathfrak{sf}}$ is the central open proj of A with $J_{\mathfrak{sf}} = \mathrm{her}(e_{\mathfrak{sf}})$, then

$$e_{\mathfrak{sf}} = \overline{e_{\mathfrak{A}}}^{e_{\mathfrak{sf}}} + e_{\mathfrak{B}}.$$

(d) $A/J_{\mathfrak{C}}^{A}$ is C^*-semi-finite and $A/(J_{\mathfrak{A}}^{A})^{\perp}$ is of type $\mathfrak{A}$.

(e) If A is C^*-semi-finite, then $A/J_{\mathfrak{B}}^{A}$ is of type $\mathfrak{A}$.

(c) $J_{\mathfrak{A}} + J_{\mathfrak{B}}$ is an essential closed ideal of $J_{\mathfrak{sf}}$.

If $e_{\mathfrak{sf}}$ is the central open proj of A with $J_{\mathfrak{sf}} = \mathrm{her}(e_{\mathfrak{sf}})$, then

$$e_{\mathfrak{sf}} = \overline{e_{\mathfrak{A}}}^{e_{\mathfrak{sf}}} + e_{\mathfrak{B}}.$$

(d) $A/J_{\mathfrak{C}}^A$ is C^*-semi-finite and $A/(J_{\mathfrak{A}}^A)^\perp$ is of type $\mathfrak{A}$.

(e) If A is C^*-semi-finite, then $A/J_{\mathfrak{B}}^A$ is of type $\mathfrak{A}$.

Papers of the speaker can be found at
http://www.math.nsysu.edu.tw/∼wong

Papers of the speaker can be found at

http://www.math.nsysu.edu.tw/∼wong

Thank you!

Papers of the speaker can be found at

http://www.math.nsysu.edu.tw/∼wong

Thank you!

A Chinese dragon story

莊子・列禦寇:

“朱泙漫學屠龍於支離益, 殫千金之家,三年技成,而無所用其巧” 。

A Chinese dragon story

莊子・列禦寇:

“朱泙漫學屠龍於支離益, 殫千金之家,三年技成,而無所用其巧”。

Quoted from Zhuang-zi (a Chinese philosopher):

“A man Zhu learned the fine skill of killing dragons from Master Chi. Has spent all his money, he became a master in three years. However, he then found no dragon at all in the world.”

A Chinese dragon story

莊子・列禦寇:

“朱泙漫學屠龍於支離益, 殫千金之家,三年技成,而無所用其巧”。

Quoted from Zhuang-zi (a Chinese philosopher):

“A man Zhu learned the fine skill of killing dragons from Master Chi. Has spent all his money, he became a master in three years. However, he then found no dragon at all in the world.”